2.10 Exponential Integrals Ei and E_1

A. Purpose

These subroutines compute the exponential integrals Ei and E_1, defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt, \quad \text{and} \quad E_1(x) = \int_{x}^{\infty} \frac{e^{-t}}{t}\,dt.$$

These functions are related by the equation

$$\mathrm{Ei}(x) = -E_1(-x)$$

The functions $\mathrm{Ei}(x)$ for $x > 0$ and $E_1(x)$ for $x < 0$ are defined as Cauchy principal value integrals. These functions thus have well-defined finite values for all real x except $x = 0$ where $\mathrm{Ei}(0) = -\infty$ and $E_1(0) = +\infty$.

For additional properties of these functions see [1].

B. Usage

B.1 Program Prototype, Single Precision

REAL X, Y, EI, SE1

Assign a value to X and obtain the value of Ei or E_1 respectively by use of the statements,

Y =SEI(X)   **Y =SE1(X)**

B.2 Argument Definitions

X [in] Argument of function. Require X $\neq$ 0.

B.3 Modifications for Double Precision

For double precision usage change the REAL statement to DOUBLE PRECISION and change the function names to DEI and DE1 respectively.

C. Examples and Remarks

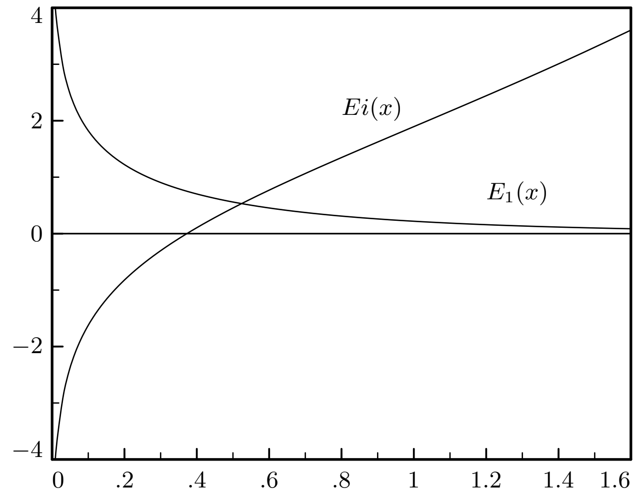
See the program DRSEI and the output ODSEI for an example of the use of SEI and SE1 to tabulate values of Ei and E_1.

D. Functional Description

As x varies from $-\infty$ to 0, $E_1(x)$ varies monotonically from $-\infty$ to $+\infty$. There is a single real root at $-0.37250\ 74107\ 81366\ 63446$.

As x varies from 0 to $+\infty$, $E_1(x)$ varies monotonically from $+\infty$ to zero.

$E_1(x)$ is asymptotic to $x^{-1}e^{-x}$ as x approaches $+\infty$ or $-\infty$, and to $-\ln|x|$ as x approaches zero.



Let μ and λ be defined so that e^{μ} is the overflow limit and $e^{-\lambda}$ is the underflow limit for the machine arithmetic. Define $\alpha = \mu + \ln\mu$ and $\beta = \lambda - \ln\lambda$. Then $E_1(x)$ would overflow for $x < -\alpha$ and underflow for $x > \beta$.

This algorithm, due to L. W. Fullerton, with minor changes by Lawson and Chiu, partitions the interval $[-\alpha, \beta]$ into eight subintervals. On each subinterval a polynomial approximation is used.

The polynomial degrees and the numbers α and β are determined on the first entry to the subprogram by use of the System Parameters subprograms (see Chapter 19.1). The subprograms adapt to any precision up to about 31 decimal places.

Accuracy tests

Subprogram SE1 was tested on an IBM compatible PC using IEEE arithmetic by comparison with DE1 at 50,000 points between −80 and 80. The relative precision of the IEEE single precision arithmetic is $\rho = 2^{-23} \approx 1.19 \times 10^{-7}$. The test results may be summarized as follows:

Argument Interval	Max. Rel. Error
[−80.00, −1.20]	2.5ρ
[−1.20, −1.00]	4.6ρ
[−1.00, −0.41]	0.9ρ
[−0.41, −0.30]	(see just below)
[−0.30, 80.00]	0.8ρ

The relative error in the interval $[-0.41, -0.30]$ is large due to the root near -0.3725. However, $|E_1(x)|$ is bounded by 0.31 and the absolute error has a satisfactorily small bound of 0.22ρ in this interval.

References

1. Milton Abramowitz and Irene A. Stegun, **Handbook of Mathematical Functions**, *Applied Mathematics Series 55*, National Bureau of Standards (1966) Chapter 5, 227–252.

E. Error Procedures and Restrictions

In the following cases the function value would be beyond the representable range. The subprograms will issue an error message and return a value as follows (Ω is the overflow limit):

X	SEI(X)	X	SE1(X)
$< -\beta$	0	$< -\alpha$	$-\Omega$
$=$ 0.	$-\Omega$	$=$ 0.	Ω
$>$ α	Ω	$>$ β	0

Error messages are processed using the subroutines of Chapter 19.2 with an error level of zero.

F. Supporting Information

The source language is Fortran 77.

Based on code designed and programmed by L. W. Fullerton, Los Alamos National Lab., 1977. Modified by C. L. Lawson and S. Y. Chiu, JPL, 1983.

Entry	Required Files
DE1	AMACH, DCSEVL, DEI, DERM1, DERV1, DINITS, ERFIN, ERMSG, IERM1, IERV1
DEI	AMACH, DCSEVL, DEI, DERM1, DERV1, DINITS, ERFIN, ERMSG, IERM1, IERV1
SE1	AMACH, ERFIN, ERMSG, IERM1, IERV1, SCSEVL, SEI, SERM1, SERV1, SINITS
SEI	AMACH, ERFIN, ERMSG, IERM1, IERV1, SCSEVL, SEI, SERM1, SERV1 SINITS

DRSEI

```
      program DRSEI
c>> 1996-05-28 DRSEI   Krogh Added external statement.
c>> 1994-10-19 DRSEI   Krogh  Changes to use M77CON
c>> 1992-03-16 DRSEI   CLL
c>> 1990-11-29 CLL
c>> 1987-12-09 DRSEI   Lawson  Initial Code.
c     ------------------------------------------------------------------
c--S replaces "?": DR?EI, ?EI, ?E1
c     ------------------------------------------------------------------
      integer J
      external SEI, SE1
      real          X(14), Y, Z, SEI, SE1
c     ------------------------------------------------------------------
      data X / -80.E0, -20.E0, -5.E0, -1.E0, -.4E0, -.3E0, -.001E0,
     *         .001E0, .3E0, .4E0, 1.E0, 5.E0, 20.E0, 80.E0 /
c
      print '(1x,3X,A1,13X,A6,14X,A6/)', 'X', 'SEI(X)', 'SE1(X)'
c
      do 10 J = 1, 14
        Y = SEI(X(J))
        Z = SE1(X(J))
        print '(1x,F7.3,5X,2(G15.8,5X))',X(J),Y,Z
   10 continue
      end
```

ODSEI

X	SEI(X)	SE1(X)
−80.000	−0.22285430E−36	−0.70145861E+33
−20.000	−0.98355261E−10	−25615652.
−5.000	−0.11482956E−02	−40.185276
−1.000	−0.21938396	−1.8951187
−0.400	−0.70238012	−0.10476526
−0.300	−0.90567666	0.30266854
−0.001	−6.3315392	6.3295393
0.001	−6.3295393	6.3315392
0.300	−0.30266854	0.90567666
0.400	0.10476526	0.70238012
1.000	1.8951187	0.21938396
5.000	40.185276	0.11482956E−02
20.000	25615652.	0.98355261E−10
80.000	0.70145861E+33	0.22285430E−36